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A NEW PROOF OF THE GITIK–SHELAH THEOREM

BY

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ABSTRACT

A purely combinatorial proof of a recent result of M. Gitik and S. Shelah is presented.

In this note we present a purely combinatorial proof of a recent result of M. Gitik and S. Shelah. For a cardinal λ, let 2^λ be the usual product space. Denote by $\mathbb{B}_\lambda$ the σ-field generated by the basic open sets in 2^λ. Let $\mathbb{K}_\lambda$ and $\mathbb{L}_\lambda$ denote, respectively, the ideals of meagre and measure zero sets in 2^λ.

THEOREM (Gitik, Shelah [GS]). *Let I be a nonprincipal, κ-complete ideal on a regular, uncountable cardinal κ. If for some $\lambda \geq \aleph_0$ the Boolean algebra $P(\kappa)/I$ is isomorphic either to $\mathbb{B}_\lambda/\mathbb{K}_\lambda$ or to $\mathbb{B}_\lambda/\mathbb{L}_\lambda$, then $\lambda \geq \kappa^+$.*

Our formulation is slightly different from that of [GS] (compare Corollary 1.4 and Theorem 2.6 in [GS]). The proof in [GS] uses metamathematical techniques like forcing and generic ultrapowers. I have received a letter from Professor D. H. Fremlin with another combinatorial proof of the Gitik–Shelah theorem.

§0. Preliminaries

We fix the notation and recall some definitions. **Ord** is the class of all ordinal numbers in the sense of von Neumann.

Let I be an ideal on cardinal κ. We consider only nontrivial ideals, i.e. $\varnothing \in I$ and $\kappa \notin I$. I is *nonprincipal* iff $[\kappa]^{<\omega} \subseteq I$; I is *uniform* iff $[\kappa]^{<\kappa} \subseteq I$; I is *κ-complete* iff $\bigcup \mathfrak{A} \in I$ for each $\mathfrak{A} \subseteq I$ such that $|\mathfrak{A}| < \kappa$; I is a *σ-ideal* iff I is $\aleph_1$-complete. We let $I^+ = \{A \subseteq X : A \notin I\}$ and $I^c = \{A \subseteq X : X \setminus A \in I\}$.

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Let X be an arbitrary set. For $D \subseteq \kappa \times X$, $\alpha < \kappa$ and $x \in X$ we let $D_\alpha = \{x \in X : \langle \alpha, x\rangle \in D\}$ and $D^x = \{\alpha < \kappa : \langle \alpha, x\rangle \in D\}$.

We sometimes abuse the above notation by treating a κ-sequence $\langle D_\alpha : \alpha < \kappa\rangle$ of subsets of X as a single $D \subseteq \kappa \times X$.

Assume that I is an ideal on κ and J is an ideal on X. We define two ideals on $\kappa \times X$. For $D \subseteq \kappa \times X$:

$$D \in I \times J \qquad \text{iff } \{\alpha < \kappa : D_\alpha \in J^+\} \in I,$$

$$D \in (J \times I)^T \qquad \text{iff } \{x \in X : D^x \in I^+\} \in J.$$

For $S \subseteq \lambda$ we have a continuous projection $\pi_S : 2^\lambda \to 2^S$ given by $\pi_S(x) = x|S$. If $B \in \mathbb{B}_\lambda$ then there exist a countable set $S \subseteq \lambda$ and a Borel set $A \subseteq 2^S$ such that $B = \pi_S^{-1}[A]$. We call any such S a *support* of B. Similarly, if $T : 2^\lambda \to 2^\omega$ is $\mathbb{B}_\lambda$-measurable, then there exist a countable set $S \subseteq \lambda$ and a Borel function $\dot{T} : 2^S \to 2^\omega$ such that $T(x) = \dot{T}(x|S)$ for every $x \in 2^\lambda$. We call S a *support* of T.

We shall say that an ideal J on 2^λ *has base in* $\mathbb{B}_\lambda$ if for each $A \in J$ there is some $B \in J \cap \mathbb{B}_\lambda$ such that $A \subseteq B$.

$\mathbb{K}_\lambda$ and $\mathbb{L}_\lambda$ are the σ-ideals with base in $\mathbb{B}_\lambda$.

For a σ-ideal with base in $\mathbb{B}_\lambda$ and $S \subseteq \lambda$ we let

$$J_S = \{A \subseteq 2^S : \text{for some } B \in \mathbb{B}_S,\ A \subseteq B \text{ and } \pi_S^{-1}[B] \in J\}.$$

Then J_S is a σ-ideal on 2^S with base in $\mathbb{B}_S$.

By $\mathrm{ID}(\kappa, I)$ we denote the assertion that κ is a regular, uncountable cardinal and that I is a nonprincipal, κ-complete ideal on κ. We define three properties:

$$\mathrm{QMC}(\kappa, I) \text{ iff } \mathrm{ID}(\kappa, I) \text{ and } P(\kappa)/I \text{ satisfies the c.c.c.}$$

$$\mathrm{BTC}(\kappa, I) \text{ iff } \mathrm{ID}(\kappa, I) \text{ and for some } \lambda,\ P(\kappa)/I \cong \mathbb{B}_\lambda/\mathbb{K}_\lambda.$$

$$\mathrm{RVM}(\kappa, I) \text{ iff } \mathrm{ID}(\kappa, I) \text{ and for some } \lambda,\ P(\kappa)/I \cong \mathbb{B}_\lambda/\mathbb{L}_\lambda.$$

Let us briefly sketch the main idea of the proof of Theorem. Let I be an ideal satisfying the assumptions of Theorem. We look for ideals J satisfying the following *Product Property*: $I \times J \subseteq (J \times I)^T$. In §1 we show that this property holds for the ideals of meagre and measure zero sets. In §2 we prove that if J satisfies the Product Property then there are no large, J-almost disjoint families of functions. On the other hand, in §3 we produce some large ADF from the assumption that $\lambda \leq \kappa$.

§1. Product property for category and measure

LEMMA 1. *The following are equivalent*:

(1) $I \times J \subseteq (J \times I)^T$.

(2) $\{\emptyset\} \times J \subseteq (J \times I)^T$.

(3) *If* $Y \in J^+$ *and* $\{C_x : x \in Y\} \subseteq I^+$ *then there exists* $Z \subseteq Y$ *such that* $Z \in J^+$ *and* $\bigcap_{x \in Z} C_x \neq \emptyset$.

PROOF. An easy verification. □

PROPOSITION 2. *Assume* BTC(κ, I). *Then* $I \times \mathbb{K}_\omega \subseteq (\mathbb{K}_\omega \times I)^T$.

PROOF. Assume that for some λ, $P(\kappa)/I \cong \mathbb{B}_\lambda/\mathbb{K}_\lambda$.
It is easy to observe that it suffices to prove the following:

If $G \subseteq \kappa \times 2^\omega$ *is such that for* $\alpha < \kappa$, G_α *is dense open in* 2^ω, *then the set* $\{x \in 2^\omega : G^x \in I^c\}$ *is a dense* $\mathcal{G}_\delta$ *in* 2^ω.

Let $\{I_n : n < \omega\}$ be an enumeration of all nonempty, basic open sets in 2^ω. Set $K_n = \{\alpha < \kappa : I_n \subseteq G_\alpha\}$. Then $G = \bigcup_{n<\omega} K_n \times I_n$. For $x \in 2^\omega$ we have $G^x = \bigcup\{K_n : n < \omega \text{ and } x \in I_n\}$.

Consider the set $\{K_n/I : n < \omega\} \subseteq P(\kappa)/I$. As is known, each countably generated subalgebra of $\mathbb{B}_\lambda/\mathbb{K}_\lambda$ has a countable dense set. Hence, there exists a family $\{A_m : m < \omega\} \subseteq I^+$ such that for all $x \in 2^\omega$ we have: $\sum\{K_n/I : n < \omega \text{ and } x \in I_n\} = 1$ iff for all $m < \omega$ there exists $n < \omega$ such that $x \in I_n$ and $K_n \cap A_m \in I^+$.

For $m < \omega$ let $H_m = \bigcup\{I_n : n < \omega \text{ and } K_n \cap A_m \in I^+\}$. Then

$$\{x \in 2^\omega : G^x \in I^c\} = \bigcap_{m<\omega} H_m.$$

So we complete the proof if we show that each H_m is dense in 2^ω. Fix $m < \omega$ and let J be an arbitrary, nonempty, open set. We shall show that $J \cap H_m \neq \emptyset$. As each G_α is dense open, we have

$$\begin{aligned}\kappa &= \{\alpha < \kappa : G_\alpha \cap J \neq \emptyset\} \\ &= \bigcup_{n<\omega} \{\alpha < \kappa : I_n \subseteq G_\alpha \cap J\} \\ &= \bigcup\{K_n : n < \omega \text{ and } I_n \subseteq J\}.\end{aligned}$$

But $A_m \in I^+$. Hence, for some $n < \omega$ with $I_n \subseteq J$ we must have $K_n \cap A_m \in I^+$. So $I_n \subseteq J \cap H_m$ and $J \cap H_m \neq \emptyset$ as required. □

PROPOSITION 3. (Kunen) *Assume* $\mathrm{RVM}(\kappa,I)$. *Then* $I \times \mathbb{L}_\omega \subseteq (\mathbb{L}_\omega \times I)^T$.

PROOF. By the assumption there exists a nontrivial, κ-additive measure $m: P(\kappa) \to [0,1]$ such that $I = \{A \subseteq \kappa : m(A) = 0\}$. Let λ be the Lebesgue measure on 2^ω and assume that $D \subseteq \kappa \times 2^\omega$ is such that for $\alpha < \kappa$, $\lambda(D_\alpha) = 0$. For $m < \omega$ we can find a set $G(m) \subseteq \kappa \times 2^\omega$ such that for $\alpha < \kappa$, $G(m)_\alpha$ is an open set of Lebesgue measure $\leq 1/(m+1)$ and $D_\alpha \subseteq G(m)_\alpha$. Let $H = \bigcap_{m<\omega} G(m)$. Denote by $m \times \lambda$ the product measure defined on the σ-field $P(\kappa) \times \mathbb{B}_\omega$.

First observe that $H \in P(\kappa) \times \mathbb{B}_\omega$. For let $\{I_n : n < \omega\}$ be an enumeration of all basic open sets in 2^ω. For $m, n < \omega$ let $K_n^m = \{\alpha < \kappa : I_n \subseteq G(m)_\alpha\}$. Then

$$H = \bigcap_{m<\omega} \bigcup_{n<\omega} K_n^m \times I_n.$$

By the Fubini Theorem $(m \times \lambda)(H) = 0$ and $\lambda(\{x \in 2^\omega : m(H^x) > 0\}) = 0$. Hence, $\{x \in 2^\omega : D^x \in I^+\} \in \mathbb{L}_\omega$ as required. □

The next two lemmas allow us to extend Propositions 2 and 3 to the spaces 2^μ for infinite $\mu < \kappa$.

LEMMA 4. *Assume* $\mathrm{QMC}(\kappa,I)$ *and* $\omega \leq \mu < \kappa$. *Let* $\{S_\alpha : \alpha < \kappa\} \subseteq [\mu]^\omega$. *Then there exists* $S \in [\mu]^\omega$ *such that* $\{\alpha < \kappa : S_\alpha \subseteq S\} \in I^c$.

PROOF. Enumerate each S_α as $\{\beta_n^\alpha : n < \omega\}$. For each n $< \omega$ the function $\alpha \mapsto \beta_n^\alpha$ splits κ into $\leq \mu$ parts. By the c.c.c. and κ-completeness of I, there is $T_n \in [\mu]^\omega$ such that $\{\alpha < \kappa : \beta_n^\alpha \in T_n\} \in I^c$. Now set $S = \bigcup_{n<\omega} T_n$. □

LEMMA 5. *Assume* $\mathrm{QMC}(\kappa,I)$ *and let J be a σ-ideal on* 2^κ *with base in* $\mathbb{B}_\kappa$. *If for each countable* $S \subseteq \kappa$, $I \times J_S \subseteq (J_S \times I)^T$ *then, for each infinite* $\mu < \kappa$, $I \times J_\mu \subseteq (J_\mu \times I)^T$.

PROOF. Routine. Use the fact that each set from $\mathbb{B}_\mu$ has a countable support and apply Lemma 4. □

COROLLARY 6. (1) *Assume* $\mathrm{BTC}(\kappa,I)$ *and* $\omega \leq \mu < \kappa$. *Then* $I \times \mathbb{K}_\mu \subseteq (\mathbb{K}_\mu \times I)^T$.
(2) Assume $\mathrm{RVM}(\kappa,I)$ *and* $\omega \leq \mu < \kappa$. *Then* $I \times \mathbb{L}_\mu \subseteq (\mathbb{L}_\mu \times I)^T$. □

We cannot have $\mu = \kappa$ in Corollary 6 because of the following counterexample: Let $\{X_\alpha : \alpha < \kappa\}$ be any partition of κ such that $|X_\alpha| = \aleph_0$ for $\alpha < \kappa$. For $\alpha < \kappa$ we let

$$B_\alpha = \{z \in 2^\kappa : (\forall \beta \in X_\alpha)(z(\beta) = 0)\}.$$

Then $B_\alpha \in \mathbb{B}_\kappa$ and B_α is closed, nowhere dense and of measure zero. Let $H = \{z \in 2^\kappa : (\exists\alpha < \kappa)(\forall\beta > \alpha)(z(\beta) = 0)\}$. Observe that if $z \in H$ then $z \in B_\alpha$ for all but $< \kappa$ α's. By the uniformity of I,

$$H \subseteq \{z \in 2^\kappa : B^z \in I^c\} \subseteq \{z \in 2^\kappa : B^z \in I^+\}.$$

On the other hand, if $S \in [\kappa]^\omega$, $D \subseteq 2^S$ and $D \neq 2^S$ then H is not contained in $\{z \in 2^\kappa : z|S \in D\}$. This implies that $H \notin \mathbb{K}_\kappa \cup \mathbb{L}_\kappa$.

§2. Product property implies that there are no large ADF

To the end of this paragraph we assume that I is an ideal on κ and J is an ideal on some set X.

LEMMA 7. *Assume that I is uniform and $I \times J \subseteq (J \times I)^T$. Let $\{z_\beta : \beta < \kappa\} \subseteq X$. If $A \subseteq \kappa$ and $\{z_\beta : \beta \in A\} \in J^+$, then for some $\alpha < \kappa$, $\{z_\beta : \beta \in A \cap \alpha\} \in J^+$.*

PROOF. Assume otherwise. For $\alpha < \kappa$ let $D_\alpha = \{z_\beta : \beta \in A \cap \alpha\}$. Then $D \in I \times J$. Since D_α's are increasing, for $\beta \in A$, z_β belongs to almost all D_α's. Hence $\{z_\beta : \beta \in A\} \subseteq \{x \in X : |\kappa \backslash D^x| < \kappa\}$. By uniformity, $\{x \in X : |\kappa \backslash D^x| < \kappa\} \subseteq \{x \in X : D^x \in I^+\}$. But this contradicts that $D \in (J \times I)^T$. □

LEMMA 8. *Assume that $|X| < \kappa$ and that I is κ-complete. Then $(J \times I)^T \subseteq I \times J$.*

PROOF. Assume that $D \in (\{\varnothing\} \times I)^T$. Hence, for every $x \in X$, $D^x \in I$. As $|X| < \kappa$ and I is κ-complete, we have $\bigcup_{x \in X} D^x \in I$. But obviously

$$\{\alpha < \kappa : D_\alpha \in J^+\} \subseteq \{\alpha < \kappa : D_\alpha \neq \varnothing\} \subseteq \bigcup_{x \in X} D^x.$$

Hence D ∈ I × J. □

DEFINITION. A family $\mathfrak{F}$ of functions on X is said to be *J-almost disjoint* iff for each distinct $T, T' \in \mathfrak{F}$, $\{x \in X : T(x) = T'(x)\} \in J$.

If $\mathfrak{F} \subseteq {}^X\mathbf{Ord}$ then we shall say that $\mathfrak{F}$ is *(J, κ, κ)-concentrated* iff there exists a $Y \in J^+$, $|Y| \leq \kappa$ such that $\{x \in Y : T(x) < \kappa\} \in J^+$ for all $T \in \mathfrak{F}$. This may be viewed as a generalization of $\mathfrak{F} \subseteq {}^\kappa\kappa$.

PROPOSITION 9. *Assume that I is nonprincipal, κ-complete and $I \times J \subseteq (J \times I)^T$. If $\mathfrak{F} \subseteq {}^X$ **Ord** is J-almost disjoint and (J, κ, κ)-concentrated family, then $|\mathfrak{F}| \leq \kappa$.*

PROOF. Assume to the contrary that $|\mathfrak{F}| > \kappa$. Let $Y \in J^+$ be such that $|Y| \leq \kappa$ and for all $T \in \mathfrak{F}$, $\{x \in Y : T(x) < \kappa\} \in J^+$. By the assumptions on I, κ is a regular cardinal. Enumerating Y, using Lemma 7 and shrinking $\mathfrak{F}$ if necessary, we see that we may assume without loss of generality that $|Y| < \kappa$.

For $T \in \mathfrak{F}$, $\kappa \cap T[Y]$ is bounded in κ. Shrinking $\mathfrak{F}$ again, we may pick $\eta < \kappa$ such that for $T \in \mathfrak{F}$, $\kappa \cap T[Y] \subseteq \eta$, hence

$$\{x \in Y : T(x) < \eta\} = \{x \in Y : T(x) < \kappa\} \in J^+.$$

From now on we fix distinct $T_\alpha \in \mathfrak{F}$, $\alpha < \kappa$.

For $\alpha < \kappa$ we let $D_\alpha = \{x \in Y : T_\alpha(x) < \eta\}$. Then $D \notin I \times J$. Using Lemma 8 for the ideal $\{Z \subseteq Y : Z \in J\}$, we obtain that

$$W = \{x \in Y : \{\alpha < \kappa : T_\alpha(x) < \eta\} \in I^+\} \in J^+.$$

The κ-completeness of I allows us to define a function $h : W \to \eta$ such that for $x \in W$, $\{\alpha < \kappa : T_\alpha(x) = h(x)\} \in I^+$.

As I is nonprincipal we can apply Lemma 1.3 twice and obtain $Z \subseteq W$, $Z \in J^+$ such that $|\bigcap_{x \in Z} \{\alpha < \kappa : T_\alpha(x) = h(x)\}| \geq 2$. This gives us two distinct $\alpha, \alpha' < \kappa$ such that $Z \subseteq \{x \in X : T_\alpha(x) = T_{\alpha'}(x)\}$. But this gives a contradiction, since $\mathfrak{F}$ is J-almost disjoint. □

§3. $\lambda \leq \kappa$ implies the existence of large ADF

To the end of this section we assume that J is a σ-ideal on 2^λ with base in $\mathbb{B}_\lambda$. The next lemma is essentially a theorem of Sikorski on inducing σ-homomorphisms applied to our case.

Lemma 10. *Assume that I is a σ-ideal on κ and $\psi : \mathbb{B}_\lambda / J \to P(\kappa)/I$ is an isomorphism. Then*:

(1) *There exists a function $R : \kappa \to 2^\lambda$ such that ψ is induced by R, i.e. for each $B \in \mathbb{B}_\lambda$, $\psi(B/J) = R^{-1}[B]/I$.*
(2) *If $A \in I^+$ then $\{R(\beta) : \beta \in A\} \in J^+$; in particular, for each $S \subseteq \lambda$, $\{R(\beta)|S : \beta \in A\} \in J_S^+$.*
(3) *If $f : \kappa \to 2^\omega$, then there exists a $\mathbb{B}_\lambda$-measurable function $T_f : 2^\lambda \to 2^\omega$ such that $\{\beta < \kappa : T_f(R(\beta)) = f(\beta)\} \in I^c$.*
(4) *If $f, g : \kappa \to 2^\omega$ and $\{\beta < \kappa : f(\beta) = g(\beta)\} \in I$ then*

$$\{x \in 2^\lambda : T_f(x) = T_g(x)\} \in J.$$

Proof. We only sketch the proof, leaving some details to the reader. First observe that if $\lambda < \omega$, then $J = \{\varnothing\}$ and the Lemma follows by easy cardinal arguments. So let us assume that $\lambda \geq \omega$.

(1) For $\delta < \lambda$ pick sets $E_\delta \subseteq \kappa$ such that $\psi(\{x \in 2^\lambda : x(\delta) = 1\}/J) = E_\delta/I$. Define $R : \kappa \to 2^\lambda$ as follows:

$$R(\beta)(\delta) = 1 \quad \text{iff } \beta \in E_\delta, \quad \text{for } \beta < \kappa \quad \text{and} \quad \delta < \lambda.$$

Using the σ-completeness of ideals, it is easy to prove by induction on the complexity of B that ψ is induced by R.

(2) This follows from (1) and the definition of J_S.

(3) For $n < \omega$ pick sets $B_n \in \mathbb{B}_\lambda$ such that

$$\psi(B_n/J) = \{\beta < \kappa : f(\beta)(n) = 1\}/I.$$

Define $T_f : 2^\lambda \to 2^\omega$ by the formula: $T_f(x)(n) = 1$ iff $x \in B_n$, for $x \in 2^\lambda$ and $n < \omega$.

(4) We have $\{x \in 2^\lambda : T_f(x) = T_g(x)\} \in \mathbb{B}_\lambda$. Now use (1) and (3). □

Proposition 11. *Assume that I is a uniform σ-ideal on $\kappa \le 2^{\aleph_0}$ and $P(\kappa)/I \cong \mathbb{B}_\lambda/J$ for some $\lambda \le \kappa$. Then there exist $\mu < \kappa$ ($\mu \le \lambda$) and $\mathfrak{F} \subseteq {}^{2^\mu}\mathbf{Ord}$ such that $|\mathfrak{F}| > \kappa$ and $\mathfrak{F}$ is a J_μ-almost disjoint, (J_μ, κ, κ)-concentrated family.*

Proof. As $\kappa \le 2^{\aleph_0}$ we fix $E \subseteq 2^\omega$ so that $|E| = \kappa$.

It is well known that there exists a family $\mathcal{G} \subseteq {}^\kappa E$ which is $[\kappa]^{<\kappa}$-almost disjoint and such that $|\mathcal{G}| > \kappa$.

Fix an isomorphism $\psi : \mathbb{B}_\lambda/J \to P(\kappa)/I$ and let $R : \kappa \to 2^\lambda$ be a function given by Lemma 10(1). Using Lemma 10(3), for each $f \in \mathcal{G}$ we pick a $\mathbb{B}_\lambda$-measurable function $T_f : 2^\lambda \to 2^\omega$ such that $\{\beta < \kappa : T_f(R(\beta)) = f(\beta)\} \in I^c$. We also pick a countable $S_f \subseteq \lambda$ such that S_f is a support of T_f. By the assumption on I, the cofinality of κ is uncountable, so each S_f is bounded in κ. Hence, by shrinking $\mathcal{G}$, we can find $\mu < \kappa$ ($\mu \le \lambda$) such that $S_f \subseteq \mu$ for each $f \in \mathcal{G}$. Now pick functions $\dot{T}_f : 2^\mu \to 2^\omega$ such that $T_f(x) = \dot{T}_f(x|\mu)$ for $x \in 2^\lambda$. Set $\mathfrak{F} = \{\dot{T}_f : f \in \mathcal{G}\}$.

We claim that $\mathfrak{F}$ has the required properties. By the uniformity of I and Lemma 10(4), we see that $\mathfrak{F}$ is J_μ-almost disjoint and $|\mathfrak{F}| > \kappa$. Consider $Y = \{R(\beta)|\mu : \beta < \kappa\} \subseteq 2^\mu$. Then $Y \in J_\mu^+$ by Lemma 10(2) and for $f \in \mathcal{G}$, $\{R(\beta)|\mu : \beta < \kappa$ and $\dot{T}_f(R(\beta)|\mu) \in E\} \in J_\mu^+$.

Now we may treat 2^ω as an ordinal number with the set E coinciding with κ. Hence, $\{x \in Y : \dot{T}_f(x) < \kappa\} \in J_\mu^+$. Thus $\mathfrak{F}$ is (J_μ, κ, κ)-concentrated. □

§4. Proof of Theorem

Assume to the contrary that $\lambda \le \kappa$ and let $J = \mathbb{K}_\kappa$ or $J = \mathbb{L}_\kappa$, depending on whether $P(\kappa)/I \cong \mathbb{B}_\lambda/\mathbb{K}_\lambda$ or $P(\kappa)/I \cong \mathbb{B}_\lambda/\mathbb{L}_\lambda$. By Proposition 11, there exist $\mu < \kappa$ and a family $\mathfrak{F} \subseteq {}^{2^\mu}\mathbf{Ord}$ such that $|\mathfrak{F}| > \kappa$ and $\mathfrak{F}$ is J_μ-almost disjoint and (J_μ, κ, κ)-concentrated. By Corollary 6, $I \times J_\mu \subseteq (J_\mu \times I)^T$. Hence, by Proposition 9, we conclude that $|\mathfrak{F}| \leq \kappa$; a contradiction. □

One could try to improve Theorem in various directions. As noticed in [GS] one cannot conclude that $\lambda \geq 2^{\aleph_0}$.

PROBLEM. Under the assumptions of Theorem: Is it consistent that $\lambda < 2^{\aleph_0}$ and λ is a regular cardinal?

Added in proof. The referee remarks that the answer to this problem is: Yes. Let κ be measurable and $2^{\kappa^{+\omega}} > \kappa^{+\omega+1}$. Add $\kappa^{+\omega+1}$ Cohen or random reals.

REFERENCE

[GS] M. Gitik and S. Shelah, *Forcings with ideals and simple forcing notions*, Isr. J. Math. **68** (1989), 129–160.